CHANGING NOTIONS
OF PROPORTIONALITY IN
PRE-MODERN MATHEMATICS (*)

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Introduction

The notions of ratio and proportionality change markedly from classical mathematics to the 19th century, when they achieve the form they have retained to the present day. This paper intends to chart the development of this process in the early modern period. It will show, in particular, that 16th-century algebra and the so-called *abbaco* books played a crucial role in bringing about the transformation of the classical notion of ratio.

Defined as «a short of relation in respect of size between two magnitudes of the same kind», a ratio was not a number nor a geometrical magnitude in Euclid's *Elements* (1). Ratios can be compared to one another, for two ratios whatever are always either equal to each other or one is greater than the other. This is property that numbers have, but magnitudes have not in the *Elements*-a segment cannot be compared to a plane figure, nor a plane figure to a three-dimensional magnitude. Ratios can as well be composed among themselves, and they

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yield ratios. This is another property ratios have in common with numbers. Because of these similarities, during the first half of the 18th century ratios were identified with numerical magnitudes for all practical purposes.

It is my contention that in order to understand the changing notions of ratio and proportionality in the early modern period two questions are to be answered separately. One concerns the numerical status of the objects compared through a ratio, or terms of the ratio. The second question concerns the status of ratios temselves. Ratios may be identified with numbers easily, once it has been settled that a radio is a relationship between two numbers. Probably for this reason it has been hitherto overlooked that the two difficulties were solved in separate stages. In what follows I shall contend that by the turn of the 17 century the first of the difficulties just mentioned had been overcome. Geometrical magnitudes were handled through their numerical measures for all practical purposes, and ratios were commonly understood to be relations between numerical magnitudes. In studying this development particular attention is paid to the problem of evaluating the influence of the medieval notion of *denomination* of ratios. Thanks to Pedro Núñez's algebra book it is possible to show that this concept had become fossilized and lost its virtuality as an «arithmetizing» agent by the mid 16th century. Rather, the changes overcoming ratios and proportionality in the 16th century are understandable only with reference to the social background, and particularly to the educational role of the *abbaco* books during the 15th and 16th centuries.

The paper examines first the place of the so-called *abbaco* schools in Renaissance society; next, it dwels on the notion of proportionality set forth in the *abbaco* books and in medieval treatises, and finally it studies the treatment received by ratios and proportionality in the works of Núñez, Stevin, Viète and Oughtred.

Mathematics, abbaco *schools, and Renaissance society*

Mathematics enjoyed a growing social appreciation during the Renaissance. We know, for instance, that a lecture by Luca Pacioli on Book V of the *Elements* could gather at least 94 outstanding humanists and citizens of Venice in 1508 (2). We know also that there was considerable fascination with applied mathematics. In the 15th century and in the first half of the 16th century there is hardly any reference to the

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abstract, axiomatico-deductive character of mathematics as a feature responsible for the superior certitude of mathematical truths. As a matter of fact, little more than lip-respect was paid to the great texts of classical mathematics before the first quarter of the 16th century. As early as about 1400 good Greek copies of the major classical mathematical sources, including Euclid's *Elements,* were available in the humanist libraries of Florence, Rome and Venice (3). Yet not until the 1530's and 1540's did the systematic edition of classical mathematics start (4). We know of Maurolico's motivations through his letters to Cardinal Bembo in 1536 and 1540. Making classical mathematics widely available, he says, is urgent because mathematics are most useful to «physicians, lawyers, farmers, sailors and merchants» (5). This was the favourite song played to mathematics during the 15th and 16th centuries and almost the only one until about 1550. This fascination with the practical uses of mathematics is adequately conveyed by Raphael's celebrated *School of Athens* (1510), where he portayed Phythagoras and Euclid as contrasting figures surrounded by two separated groups of disciples. While Pythagoras, presented as a follower of «Plato, was teaching harmony, Euclid was teaching geometry. Presented as a follower of Aristotle, and having Raphael himself among his students, Euclid meant knowledge about physical, useful things concerning painters, architects and engineers, rather than a way to contemplate an unchallengeable truth.

As his father told the story, in the 1470's Nicolo Machiavelli attended a *scuola de grammatica* between age seven and ten. Until the age of twelve he studied with a *maestro d'abbaco,* to begin afterwards the study of Latin classics (6). Slightly shorter than the average stays for people who eventually attended university, the number of years Machiavelli spent in each educational level is yet representative of the intellectual training received by educated people in 15th-century Italy (7). As it has been shown recently, *abbaco* schools are of capital importance for the development of mathematical thought in the 15th and 16th centuries

> «The development of algebra in Italy... took place primarily within the abbacus tradition, the circle formed by the *maestri d'abbaco* and their followers, together with the treatises and problems that the composed and passed to each other» (8).

The content of the *abbaco* treatises that survive give us a sense of the mathematical knowledge proper to the *abbaco* tradition: «The bulk

of their space is.. devoted to the working out of problems, predominantly commercial problems like pricing, monetary exchange, barter, partnerships, interest, and discount», but also recreational problems. «In order to solve these problems the texts usually use... standard methods... The most important of these are the rule of three, the rules of single and double false position, and algebra» (9). The *abbaco* tradition originated in fourteenth and fifteenth century Italy thanks, particularity, to the large class of merchants, clerks, and shopkeepers engendered by Italy's central position in the economy of the period (10). Spreading from Italy, this tradition was well established and flourishing all over Europe by the turn of the 16th century, when printed editions of commercial arithmetics, the direct off springs of *abbaco* treatises, became common (11).

The *abbaco* schools had a marked urban mercantile character, a character plainly manifest in the *abbaco* books that have come down to us. Most of the problems there solved had a strong applied character, involving different instances of proportional division. Yet in these books proportionality was anything but a theoretical notion. Composition of ratios, equality of ratios, denomination of ratios, classification of ratios in *rational* and *irrational*, or in ratios of *equality* and *inequality*, notions which were very important in the classical and medieval treatment of ratios, are altogether alien to *abbaco* books. As we will see below, the medieval mathematical tradition was concerned in setting up an arithmetic of ratios. The *maestri d'abbaco*, on the other hand, were interested in proportionality only as a relationship linking three or more numbers. So, while dealing with proportionality as a series of numerical rules computationally useful, they did not use, nor study, the notion of ratio in itself. On the other hand, *Abbaco* books featured a new understanding of numbers which identifies them with measures; a new importance given to problems related to the determination of geometrical measures; and a new emphasis in the arithmetical skills needed to handle radical numbers.

The *abbaco* treatise by Piero della Francesca, famous painter and unknown but worthy mathematician, is a perfectly good example of the links between *abbaco* books and the origins of algebra. Written in the third quarter of the 15th century, della Francesca's treatise is of interest in itself for an attempt to solve algebraic equations of degree higher than two (12). It contains also the only reference I know of in modern times, prior to Simon Stevin's 1685 *Arithmetique*, to the Euclidean algorithm for the great common multiple of integers (13).

As an example of how proportionality came to treated algebraicaly we shall examine a proportionality problem solved by della Francesca. Two numbers are sought such that one is the same part of the other as 3 is of 4, their product being $10 + \sqrt{10}$. Whitour further explanation, della Francesca calls the two numbers 3x and 4x and then sets forth the ensuing equation:

> Poni che il primo numero sia 3 cose [3x] e l'altro 4 cose [3x]; montiplica 3 cose via 4 cose fa 12 censi [$12x^2$], li quali sono equale a 10 e radici de 10 (14).

Clearly there is nothing operationally new in the rules *abbaco* books used to deal with proportionality, for they were already present in the so-called arithmetical books of Euclid's *Elements* (15), Yet in practice the rules were mostly useless there, for they were applicable only to integer numbers, and not even generally. *Abbaco* books, however, had no scruple in handling fractions and radicals as if they were integers. A mathematical practitioner who followed the arithmetical rules set forth in *abbaco* books used fractions and radicals in ways that the faithful follower of Euclid was not allowed. In this way, by casting them in a new system of numerical notions, *abbaco* books gave to the old, well-known arithmetical rules governing proportinality in the *Elements* a new meaning and importance.

Most interestingly, as Baxandall pointed out, the mathematical program of the *abbaco* schools expresses itself through the new tastes of the time. The painter and mathematician Piero della Francesca, Luca Pacioli's work and ties with Leonardo da Vinci, or the relationship between Brunelleschi and the cosmographer and mathematician Paolo Toscanelli are well-know examples of the intellectual affinity between artists and mathematicians of the Italian Renaissance. Artists and patrons, Baxandall has shown, were able to agree in evaluating a perspective based picture because they shared the mathematical skills needed to understand the painting (16).

Yet it may not have been adequately emphasized till now that the Renaissance taste for the geometrical organization of visual space, and the emphasis on geometry as the basis for the visual arts in general, entails a new understanding of mathematics. In the visual space represented realistically, or perspectively organised, lengths and geometrical figures were *measured* and their *numerical* proportions

harmoniously rearranged on a plane surface (17). Preoccupation with the actual numerical proportions to be given to human figures is paramount in Dürer's geometrical sketchs of bodies, as well as in Leonardo's (18). The way in which geometrical proportions were handled in painting textbooks, as for example in della Francesca's *De prospectiva pingendi*, is also good evidence of the numerical approach —as opposed to the non-numerical approach found in Euclid's *Elements*— to geometry used by Renaissance artists.

To summarize, the mathematics taught in the *abbaco* schools, and the geometry taught in perspective treatises, conveyed an understanding of proportionality which was in important respects different from the way proportionality was treated in Euclid's *Elements*. As we shall see now, it was also different from the treatment it received in scholastic treatises, some of them still used by university teachers in the late Renaissance-for instance Jordanus Nemorarius' *Arithmetica*, which knew several editions in Paris from 1495 on. Now in order to study in technical detail the way in which ratios and proportionality changed, we need first summarize the main features of the classical and medieval tradition. This is not merely an erudite exercise, for it has been argued that internal conceptual developments taking place within this tradition opened the way to changes in the notions of ratio and proportionality.

Scholastic medieval developments

Vis-à-vis the genuine *Elements*, the more substantial variations occurrying in medieval versions of Euclid's *Elements* cluster around the notion of proportionality (19). They are, on the one hand, the modifications introduced into Definitions 4 and 5 of Book V and, on the other, the suppresion of Proposition 12 of Book VI and the addition of an spurious postulate to Book 1 (20).

Definition 5 in Book V (briefly, V-5) of Euclid's *Elements*, the genuine Eudoxian definition of equal ratios, reads thus:

> 5. Magnitudes are said to be in the same ratio, the first to the second and the third to fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order (21).

Definition V-4 merely states the condition of existence of a ratio between two magnitudes whatever:

> 4. Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another (22).

In the Adelard-Campanus version of the *Elements*, the standard Euclidean text from the middle of the 13th century until the 16th-century editions of humanist inspiration, the following statements substituted Definitions 4 and 5:

> 4'. A, B, C are continuous proportionals if and only if nA, nB, nC are so.
> 5'. A, B, C, D are proportional if and only if nA, mB, nC, mD are so (23).

These are meaningless definitions, to be sure, which show that the equality of ratios was deemed somehow known without the rules contained in the *Elements*. Evidence to support this comes form the spurious definition we find added to Book VII of the *Elements* (and even to Book V in some versions), that is the definition of the equality of ratios through the equality of their *denominations*.

Scholastic treatises dealing with ratios and proportionality introduced the notion of denomination *(denominatio)* of a ratio. Among the first occurences of this notion are those we find in Jordanus Nemorarius' work, in the first quarter of the 13th century. In Jordanus' definition the denomination of the ratio (A:B) was that which results in the division of the antecedent, A, or first term of the ratio, by the consequent, B, or second term: «Denominacio vero proporcionis huius ad illud est quod exit ex divisione huius per illud» (24). Jordanus' words did *not* mean that, for instance, the ratio of 20 to 12 is the number 1.666... My anachronistic use of decimal fractions is just a way to underscore how anachronistic it is to apply to Jordanus' words their present meaning, for they clearly belong to he context of the medieval classification and naming of ratios (25). Thus, by dividing 20 by 12 we get $1 + 2/3$, which allows us to determine that the ratio of 20 to 12 is the super-bipartiens thirds (26). In general, assuming A and B commensurable and $A > B$, the division of the antecedent A by the consequent B yields an integer number plus a fraction, which in turn determines the name of (A:B). According to Murdoch, Jordanus' *De elementis arismetice*, or

Arithmetica, clearly influenced Campanus' edition of the *Elements.* Coming from Jordanus, in particular, is the definition of the equality of ratios through the equality of their denominations, which Campanus added to Book VII (27).

Campanus explicitly warned that denominations cannot be used everywhere, for only ratios between commensurable terms —in practice, between integer numbers— have denominations. However, once established in Book VII of the influential Campanus edition, the new definition of equality o ratios —that is, that ratios are equal which denominations are equal— became the 5'', generally accepted definition of equality of ratios, seeped back to Books VI and V, and contaminated the Eudoxian theory of ratios. Later versions of the *Elements,* as well as most scholastic mathematical treatises, not only took up the new definition using denominations, but also substituted it for the genuine Eudoxian definition of equality of ratios. Roger Bacon, Bradwardine and Albert of Saxony, among others, completely failed to appreciate the necessity of the original, involved definition (28).

This failure, coupled to Campanus' transformation of the genuine Definition V-5, must be construed as a failure to recognize the anumerical character of Euclid's geometry. Lines, figures and geometric bodies had no measures in the *Elements.* Quadratures and cubatures yielded a ratio or comparison between two figures or two bodies, but did not yield a number. This understanding of geometrical objects excluded *a priori* that the equality of two ratios $(A:B)=(C:D)$, be defined in general through the equality of the cross products, $A \times D = B \times C$, for these products did not exist as such. This particular characterization of the equality of ratios, however, is possible when the terms of the ratios are numbers, and Euclid did not fail to use it in Books VII, VIII, and IX, the so-called arithmetical books of the *Elements.* Dealing with positive integer numbers exclusively, these books solve, among other things, several questions concerning proportionality, such as how to determine the fourth proportional term to three given numbers. It is important to underline that similar questions had been previously solved in Books V and VI. However, the answers to these questions are different there, for they consisted of geometrical constructions (29). Classical mathematics, therefore, featured a rigid distinction between general magnitudes devoid of measure, to which apply the results on proportionality set forth in Books V and VI, and numbers, to which apply Books VII, VIII, and IX. This rigid compartmentalization between geometry and arithmetic in the *Elements* is what openly disappears in

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the 16th century, when the notions of geometrical object and number both change, as we shall see below.

That this rigid distinction had started to crumble in the 13th century is not only revealed by the failure to preserve the genuine Definition V-5, but also by Jordanus Nemorarius' failure to understand the purely arithmetical character of Books VII, VIII and IX. In one of his best known works, *De numeris datis*, Jordanus deals with proportionality in a way that very much illuminates this point (30). He included there, without demonstration, that the cross products of four numbers are equal when the numbers are proportional, and then grounded on this result the so-called *golden rule* for finding the fourth proportional to three given numbers, that is, that $x = ab/c$ is the fourth proportional to a, b, and c (31). Proposition 1 from Book III uses the same argument to deduce that $x^2 = \sqrt{ab}$ is the mean proportional to the numbers a and b. Again, Jordanus was not offering new results here, for the arithmetical books of the *Elements* contain all of these results, and many others (32). What sets Jordanus' work apart from the *Elements* is that the *Elements* clearly state and prove *conditions of existence* for the proportional means obtained through rules involving arithmetical operations (33). Conditions of existence are important to Euclid because arithmetical operations will not give us integer numbers necessarily. This is particularly the case with roots, but even the fourth proportional, $x = ab/c$, may not exist if ab is not divisible by c.

Jordanus was not concerned by the conditions of existence Euclid's rules required to ensure their applicability, which probably indicates he did not take over the strong Euclidean requirement that operations between positive integer numbers are only acceptable if they yield positive integer numbers. On the other hand, and this sets Jordanus apart from the 16th-century algebrists, he did not work with a novel notion of number broad enough to comprehend radical quantities as well as integers and fractions. To be sure, we do not find such quantities used anywhere in Jordanus' works. In *De numeris datis*, in particular, all the numbers involved in computations of proportional terms are well chosen enough as to yield always integer solutions (34).

Scholastic books dealing with ratios considered irrational numbers only ocassionally, usually in the context of making it clear that the author knew the existence of ratios other than those between integer numbers. As far as I know, the notion of denomination of a ratio between integer numbers or commensurable magnitudes was never ap-

plied as such to irrational ratios. No scholastic mathematician, for instance, took the denomination of the ratio between the diagonal of the square and its side, or $(2:1)^{1/2}$, to be the number $\sqrt{2}$, which results from assuming the antecedent to be $\sqrt{2}$ and then dividing the antecedent by the consequent. While the denomination of a rational ratio —that is, one between integer numbers— is found through a simple division, irrational ratios were given denominations of a different kind.

Nicolas Oresme's account of the denomination of irrational ratios is the fullest we know of (35). Oresme made extensive use of denominations of ratios, both rational and irrational, in order to compare them and decide about their *commensurability in the exponential sense*. In modern terms, Oresme called two ratios **A** and **B** *commensurable* when two integer numbers *m, n* exist such that $A = B^{m/n}$. In order to see how Oresme introduced the distinction between mediate and inmediate denominations we shall deal with the ratio the diagonal of the square bears to the side, or $(2:1)^{1/2}$. According to Grant —but the matter has received other interpretations, since Oresme's ideas were imperfectly expressed— this ratio was *immediately* denominated by the radio (2:1) and *mediately* denominated by the number 1/2 (36). Medieval mathematicians often used additive language to express the composition of ratios. Thus, for instance, the ratio $(A:B)^2$ was called a ratio *double* of (A:B), and (A:B) was *one third* of $(A:B)^3$. Two commensurable (in Oresme's sense) ratios **A** and **B** were said to *bear the ratio* (m:n), if $A = B^{m/n}$. The very title of Oresme's important treatise, *De proportionibus proportionum*, makes reference to this peculiar understanding of ratios between ratios (37). One of the main topics in Oresme's treatise is the study of the conditions under which ratios of ratios existe (38). In order to do so, first Oresme studied ratios (exponents, that is) between ratios among integer numbers, or rational ratios. He demostrated results such as, «No multiple ratio is commensurable to any greater non-multiple ratio» (39). On these results Oresme based his study of commensurability of ratios in general and here is where mediate and immediate denominations are important to Oresme. In order to know, for instance, whether the ratio $A = (2:1)^{1/2}$ is commensurable with the ratio **B**, then the ratio **C** immediately denominating **B** is needed; for instance, if $B = (4:1)^3$, the C = (4:1). Once **C** is compared to the ratio immediately denominating **A**, (2:1) in this case, a conclusion is reached about the commensurability of **A** and **B** (40).

In Oresme's treatise, as Murdoch has already pointed out, denominations were *not* used to perform operatios between ratios, or to dilucidate equality of ratios. In fact, Oresme's denominations of irrational ratios are quite useless in this respect (41).

We shall turn now to look closely at the changing way in which ratios and proportionality were handled in the second half of the 16th century. An important question is, what happened to the notion of denomination in the 15th and 16th centuries? As we shall see now, a partial answer can be found in Pedro Núñez's 1567 algebra book, which still contains resonances of the medieval theory of ratios.

Pedro Núñez (1502-1577)

Even though it does not contain innovative results comparable to those in Cardano's *Ars Magna,* nor is it so comprehensive and well organised as Stevin's *L'Arithmetique,* Núñez's 1567 *Libro de Algebra en Arithmetica y Geometria* is an excellent cossist book. As is known, Stevin was acquainted with it and credited Núñez with leading him to apply Euclid's algorithm to polynomials (42).

The *Libro de Algebra* is divided in three parts. Very short, some 20 small sheets, the first part sets forth the resolution of first and second degree equations. Very long, some 200 sheets, the last part contains a collection of arithmetical and geometrical problems. The second part, some 100 sheets, contains a careful exposition of the theory of ratios and proportionality, which occupies some 50 sheets, along with sections explaining the handling of roots and polynomials. Núñez considered proportionality a privileged subject to provide rigorous foundations to algebraic rules. That proportionality has not been placed at the beginning of the work, the reader is told, neither makes the work less rigorous, nor is the ordering of the topics inconsistent with the grounding role of proportionality (43).

Núñez's account of proportionality is not particularity innovative. Its interest for the history of mathematics comes rather from the opposite direction; it has the kind of interest fossils have. We find in Nuñez's account the topics already present in Oresme, and we find them developed with more clarity but with the same limitations. According to Núñez, the denomination of a rational ratio (A:B) of greater inequality (that is, such that A > B) is the number by which the antecedent A comprehend the consequent B or exactly or with one part or with several

parts (44). Núñez then teaches how to go from the ratio (from its *name*, in fact) to its antecedent and consequent. The name of the radio says how many times the greater term contains the lesser. For instance, given the ratio supertripartiens fourths, we know that the antecedent is more than once contained in the consequent but less than twice (because of the prefix *super*). Furthermore, *tripartiens fourths* tells us that the given ratio is the ratio of $1+3/4$ to unity. By transforming these numbers into 7/4, adds Núñez, we know that 7 and 4 are the *numbers* (sic) of the given ratio (45). To know whether two numbers are the least ones which correspond to a ratio, they should be taken as numerator and denominator of a fraction and simplified as such:

> [C]omo si fuesen denominador y numerador de un quebrado, y si no se pueden mas abreviar, diremos que ellos mismos son los mínimos de su proporción (46).

To find what the proportion is that two given numbers bear other, the greater of them must be divided by the lesser. The quotient is the denomination, which readily determines the name of the ratio, that is, the ratio itself (47).

Núñez mentions two methods to carry out the composition of ratios, the multiplication of denominations, which applies only to rational proportions (48), and the product of antecedents and consequents (49). The latter is demonstrated using that, by Euclid's Proposition V-15,
$(A:B)=(AxC:\ B\times C)$ and $(C:D)=(CxB:DxB)$.

Now, the composition of (AxC: BxC) and (CxB:DxB), because the term $B\times C$ is the consequent of the former and the antecedent of the latter, is (AxC:D$\times$B), wherefrōm the rule follows (50). It should be noticed that Núñez's proof is incomplete, because Proposition V-15 of the *Elements* states that $(A:B)=(mA:mB)$, only *if m is an integer number*.

Let us consider Núñez's understanding of denominations of irrational ratios. In page 75 [R], where denominations are first mentioned, we are told that «...le dan [a la proporción] por esta causa quantidad, y sera la quantidad de la proporción la su denominación», and then he goes on to define the denomination of a rational ratio in the terms mentioned above (51). In this context Núñez explains that the denominations of irrational ratios cannot be numbers, which according to the usage of the time means that they cannot be rational numbers. Núñez did not fail to introduce *mediate* denominations for irrational ratios:

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> Algunas de las [proporciones] irracionales son denominadas de pro-
> porciones inmediatamente, y de números mediatamente. Exemplo, la
> proporción que tiene R.2. [√2] para la unidad, es irracional, y es la
> mitad de una proporción dupla. Toma luego la denominación de la
> proporción dupla inmediatamente, y del número 2 que es denominador
> de la dupla, mediatamente.

Acoording to Núñez, the ratio $(2:1)^{1/2}$ is immediately denominated by the ratio (2:1) and mediately by 2, the 2 coming from (2:1) (52).

The important notion in the denomination of irrational ratios, for Núñez as for his medieval predecessors, was the *immediate* denomination by a ratio. This notion was operationally meaningful because allowed the authors to decide about the commensurability (in Oresme's sense) of irrational ratios. The number that denominated these ratios *mediately* was not much more than a rhetorical device, apparently.

In one respect Núñez departed markedly from the medieval treatment of ratios and proportionality: He freely used radical numbers as terms of ratios, when he explained how to deal with arithmetical problems involving proportions. This was a common feature of cossist algebras and *abacco* books. Not common was, however, Núñez's interest, equal to Stevin's, in providing a rigorous foundation for algebraic manipulations of this kind. According to Núñez, Euclid's *Elements* provided solid foundations for the arithmetical rules involved in the handling of proportionality, including those concerning radical quantities. Historically this is very interesting, for, as remarked above, the *Elements* discriminated between rules for integer numbers and rules for continuous magnitudes, a distinction which was no longer present in 16th-century algebra books.

There are two instances in which Núñez's references to Euclid are telling. In order to prove the *golden rule*, also called *rule of three*, which determines arithmetically the fourth propotional to three given numbers, he uses Book VI of the *Elements* (53). Book VI proves indeed the equivalence of (A:B)=(C:D) with a (geometrical) result that may be translated as BxC=AxD, *provided that the last equality is not understood as a numerical one* (54). This is why Book VI fails to mention the rule $D = \dfrac{BxC}{A}$ (which is Núñe's *rule of three*) as a way to determine the fourth proportional. As mentioned above, the one legitimate way to determine a fourth proportional in Book VI comes through the geometrical configuration associated to the so-called Thales theorem (55).

Núñez's second heterodox reading of Euclid is his attempt to

ground the computation of proportional means upon Book VIII of the *Elements*. After stating that the two proportional means between 2 and 5 are $\sqrt[3]{20}$ and $\sqrt[3]{50}$, he explains his procedure by a direct reference to those propositions in Book VIII in which two integer numbers x and y are determined such that (B:y)=(y:x)=(x:A), provided that A and B are two given *cubic* numbers (that is, $A=a^3$ and $B=b^3$, for two integers a,b). Now, Núñez is not applying this rule to cubic numbers, but rather to two integers whatsoever. Moreover, Núñez concludes by pointing out that the same rule holds for quantities which are not numbers, for instance $\sqrt{2}$ and $\sqrt{3}$. He warns the reader that the foregoing demonstration will not apply here, yet, he adds without further elaboration, it will be easily transformed in one that applies to such cases (56).

As said at the outset, Núñez offers us a fossilized theory of proportions with which three alien notions had been aggregated: the free use of radical quantities, the new arithmetical role played by antecedent and consequent terms of ratios, and the aritmetical treatment (that is, with the rules set forth in Books VII, VIII and IX of the *Elements*) of problems of proportionality involving geometrical magnitudes (which required the techniques and results of Books V and VI). The next important step in this development was given by an attentive reader of Núñez's *Libro de Algebra*, Simon Stevin, to whom we now turn.

Simon Stevin (1548-1620)

Stevin's (1585) *L'Arithmetique* is the most developed and comprehensive 16th-century cossist algebra. It contains a serious attempt to provide a notion of number encompassing the classical notions of integer number and geometrical magnitude (57). Stevin's *Traité des incommensurables grandeurs* also shows him committed to a program of arithmetization of Euclidean geometrical magnitudes. Yet, even though he erased conceptual distinctions between integrer numbers and radical quantities, he was not able to take ratios and proportionality out of their traditional conceptual framework and *problematique*. Thus, for instance, Stevin define the ratio between two arithmetical term as «la mutuelle habitudez selon la quantite entre deux ou plusieurs termes» (58). This common definition was followed by the classical distinction between ratios of equality and inequality, by an account of commensurability, and by another of the scholastic nomenclature.

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Stevin introduced the notion of «arithmetical term» to cover *racines* (radical numbers) and «arithmetical numbers» (rational numbers), thus granting the same logical status to radical numbers —the only non rational numbers known by then— and to rational numbers (59). While filling up the gap between rational and radical numbers, Stevin maintained a clear-cut conceptual distinction between ratios and numbers.

Two radical quantities are not always commensurable (in the ordinary sense). In Stevin's example, the numbers $\sqrt{50}$ and $\sqrt{2}$ are commensurable because 25, the quotient of 50 and 2, has a rational root, 5. In this case, the ratio between these magnitudes is the *quintupla*, or five-times-ratio (note that the ratio is not «five», but «five-times»). When two quantities, such as $\sqrt{3}$ and 2, are not commensurable, says Stevin, then it is enough to say that their ratio is the ratio of $\sqrt{3}$ to 2.

Stevin, as Núñez had done before, read Euclid in a heterodox way. To justify that radio of $\sqrt{50}$ to $\sqrt{2}$ is (5:1), Stevin reads Proposition VI-22 from the *Elements* as allowing him to say that (50:2)=(25:1) implies $(\sqrt{50}:\sqrt{2})=(\sqrt{25}:\sqrt{1})$. Proposition VI-22, which applies to rectilineal lines and figures, not to their measures, states that figures similarly described upon proportional lines are similar, and conversely, if figures similar and similarly described be proportional, the sides will be proportional. Stevin accompanies his proof and his reference to Euclid with a diagram showing squares that measure 50,2, 25, and 1. Their sides measure the radical quantities to which Euclid's proposition applies. To prove the same result for cubic roots. Stevin's diagram shows cubes, the sides of which are the cubic roots involved in Stevin's proposition. Once again, therefore, we have evidence that 16th-century mathematicians were reading in the *Elements* what Euclid had not put in it. They read the *Elements* as if geometrical lines and figures were identical with their numerical measures. This allowed them to apply results deduced in the arithmetical books to segments and figures, and results deduced in Books V and VI to any sort of numerical magnitudes.

Ratios reappear in a very different context, when Stevin explain s how to carry out «computations de raisons», and how to apply them to commercial problems. This material is no longer part of *L'Arithmetique*, but belongs rather to *La Practique d'Arithmetique*, a complement to the former, more theoretical treatise containing interest tables, the famous *Dime* (the first printed account of decimal fractions), and a series of other practical tools for merchants, gaugers, and surveyors. In *La Practique d'Arithmetique* the composition of ratios is used to solve problems about «compound» rules of three, namely problems in which

something is proportional to two variables. In this context, Stevin explains why this matter has not been included in *L'Arithmetique*. The reason is that ratios are not numbers, but rather notions that can only be handled through numbers. Their right place, therefore, is within the rest of applied arithmetical notions:

> [Q]uelcum me pourroit demander, pourquoy nous ne les avons pas mi en la precedente Arithmetique, Je luy respons, que icelles computations sont de purs nombres, & que la Raison (comme el apparoistra plus amplement a la multiplication des Raisons suyvante) n'est point nombre, ains subject, comme les autres matieres auquel s'applique le nombre, parquoy leur lieu n'y estoit pas (60).

Thus, at the end of the 16th century a leading mathematician understood the once sophisticated, basic notion of ratio to be a primary, intuitive notion, the use of which was suitably placed at the back of the book, along with techniques to solve problems on percentages and alligations. As we shall see now, the numerical handling of proportionality was very much reinforced by the advent of symbolic algebra.

François Viète (1540-1603)

François Viète's 1591 *In artem analyticem isagoge* has long been recognized as the founding text of symbolic algebra (61). Within this important text, Chapter II, «On the rules of equations and proportions», has a special importance, for it contains the first set of (unproved) regulations for the formal inderstansing of mathematical symbols.

From the very beginning Viète emphasized the key role proportionality has in his new approach to mathematics. The two last stipulations set forth in Chapter II are thus introduced: «A soverein rule, moreover, in equations and proportions, one that is of great importance throughout analysis is this», and then follows the characterization of proportionality by the product of means and extremes:

> 15. If there are three or four terms such that the product of the extremes is equal to the square of the mean or the product of the means, they are proportional. Conversely,
> 16. If there are three or four terms and the first is to the second as the second or the third is to the last, the product of the extremes will be equal to the product of the means (62).

Dealing with *species*, or general magnitudes freed of particular values, Viéte could no longer rest either on intuitively grasping the meaning of ratio and proportionality, as Stevin had done, or on the original «equimultiple» definition from Book V of the *Elements* —for this definition pressuposses the actual possibility of finding out and comparing the equimultiples mentioned in it. Viète, therefore, put the emphasis on the characterization of proportionality through the cross product of means and extremes— which we may call the algebraic characterization of proportionality.

The whole theory of equations was affected by Viéte's new understanding of proportionality. As he stressed in the sentence closing chapter II of the *Isagoge*, just after stipulations 15 and 16,

> Thus a proportion may be said to be that from which an equation is composed and an equation that into which a proportion resolves itself (63).

Proportionality and equations are bounded together in Viète's thought. He underlined the proximity between the two notions again in chapter V of the *Isagoge,* and made it to play an important role in his *Two Treatises on the Understanding and Amendement of Equations.* Viète viewed any equation, say $x^3 + ax^2 + bx = c,$ as equivalent to the proportionality.

$$x^2 + ax + b: \sqrt{c} = \sqrt{c}:x$$

As Joan Morse has convincingly shown, the equivalence between proportionality and equations plays a crucial role in supporting Viète's theory of equations and lends coherence and unity to different parts of Viète's work (64).

Viète's novel algebraic understanding of proportionality also reveals itself at the computational level. He nowhere stated that a ratio (A:B) was to be identified with the fraction A/B, nor used specific symbolism to express proportionality in general. Nevertheless, as far as the structure of the mathematical reasoning is concerned, Viète's ratios «behave» as if they were fractions. In achieving this, an essential tool was the symbolic expressions Viète casted over the rhetorical rules used in cossist algebras and *abbaco* books. Thus, for instance, in Proposition I of his *Ad logisticem speciosam notae priores*, we find the following formal examples of the rule which determines a fourth proportional («in» was used to indicate multiplication of *species*) (65):

First	Second	Third	Fourth Proportional
A	B	C	$\dfrac{\text{B in C}}{\text{A}}$
$\dfrac{\text{A quadratum}}{\text{D}}$	B	C	$\dfrac{\text{B in C in D}}{\text{A quadratum}}$
$\dfrac{\text{A cubus}}{\text{D plano}}$	$\dfrac{\text{B quadrat}}{\text{Z}}$	C	$\dfrac{\text{Bq in C in Dpl}}{\text{Z in A cubum}}$

This and the next example show Viète to be in fact shifting the basic rules in which the handling of proportionality stood. Through his emphasis on the formal rules, he made proportionality rest on its equivalence with an equation. Of course, he was able to do so because he made one new assumption, that magnitudes whatever, no matter that they represent lines, surfaces or bodies, can always be multiplied. The magnitudes thus generated would correspond to objects of more than three dimensions.

In order to see how the handling of propotionality changed through Viète's approach, let us turn to his proof that in a series of magnitudes in continued proportion the first is to the last as the square of the first is to the square of the second. In modern language,

$$\text{if } \frac{A}{B} = \frac{B}{C}, \text{ then } \frac{A^2}{B^2} = \frac{A}{C}$$

If A be the first term, says Viète, the last is $\dfrac{\text{B } quadratum}{\text{A}}$, a result that Viète grounds on the table given above. When first and last are multiplied by A, their ratio remains unchanged (this is what happens according to *Isagoge's* Chapter II, stipulation 12). Therefore, as A is to $\dfrac{\text{B } quadratum}{\text{A}}$, sois is A *quadratum* to $\dfrac{\text{A in B } quadratum}{\text{A}}$, which is B *quadratum*, obviously. In spite of the substantial lack of symbolization, it is manifest that, Viète's reasoning es extremely close to ours, when we write

$$\frac{A}{C} = \frac{A}{\dfrac{B^2}{A}} = \frac{A.A}{\dfrac{B^2.A}{A}} = \frac{A^2}{B^2}$$

Although Viète's language is not yet as economical as ours is, we are nonetheless far away from rhetorical algebra. Indeed Viète's deductions are as economical as ours are.

Vieté's algebraic understanding of proportionality can be observed to evolve further in the work of Oughtred, the founder of the 17th-century English school of mathematics. In Oughtred's book proportionality received the form that it was to have until the early 18th century.

William Oughtred (1574-1660)

William Oughtred's *Clavis mathematicae,* one of the most influential 17th-century algebra books on that side of the English Channel (66), assumes that ratios and proportionality deal exclusively with numerical magnitudes:

> If of foure numbers given, the first bee to the second, as the third to the fourth: those four are called proportionall numbers. Now the being or habitude of one number to another is found by dividing the Antecedent by the Consequent... (67).

Thanks to the reduction to the numerical field, Oughtred *deduced* results which Viète had merely stated as unproved regulations in Chapter II of his *Isagoge:* For instance,

> 2. Wherefore, if a number multiply two numbers, the products shall be proportional to the numbers multiplied (68).
> 3. ... If foure numbers be proportionall, the product of the two extreames is equall to the product of the two meanes (69).
> 4. From hence followes the Golden Rule (so called) of Proportion [*i.e.,* if «R:S::Z:A», then $A = ZS/R$] (70).

It is unnecessary to describe what is going on here. Oughtred's statements, identical with Viète's stipulations in the important chapter II of *Isagoge,* have become obvious deductions. Viète did not justify them, doubtless because he recognized the impossibility of doing so within the framework provided by the *Elements.* Yet Ougtred disregarded this impossibility by focusing on the numerical properties that the measures of mathematical objects have.

As Oughtred's work shows, the notions of ratio and fraction became very close, when a new and more numerical understanding of mathematical objects became predominant during the second half on the 16th century. Oughtred defined fractions, or *broken numbers*, as an integer number of subdivisions of an unit. This allowed him to point out that «what ration the numerator hath to the denominator, the same hath the quantity signified to an unit». Then, as an example, he wrote that: 4:5:: $\frac{4}{5}$:1, and that R:S:: $\frac{R}{S}$:1. From this he was able to draw an importan conclusion, that (A:B)=C:D) if and only if $\frac{A}{B} = \frac{C}{D}$. In his words:

> Wherefore the termes of equal parts, or fractions, are proportional. [A]nd contrariwise (71).

Oughtred's work also shows that ratios and fractions did not merge in the early 17th century. In spite of the numerical understanding of mathematical objects. Oughtred always treated ratios and fractions as separate, well differentiated notions. In fact he used specific, different notations for each one of these notions-a practice mathematicians followed through the 17th century generally (72).

To Oughtred we are indebted for the modern notation, a:b::c:d (73). Independently of each other, and of Oughtred's, two notations for ratios and proportionality appeared on the continent in the 1630's, though only one reached wide diffusion. Herigone represented the proportionality of HG, GA, HB, and BD by

hg π ga 2/2 hb π bd.

(One have to keep in mind that π was not used as yet to represent the ratio of the circumference to the radius, and that «2/2» was Herigone's symbol for equality, while «2/3» consistently stood for our « < », and «3/2» for our « > ») (74). The second, more widely used notation comes from Descartes, who used it in his notes of 1619-1621. There we find 1/2//4/8//16/32 for Oughtred's 1:2::4:8::16:32 (75). With minor variations Descartes notation was used, but not predominantly, until early in the 18th century (76).

The 17th-century symbolical usages here briefly sumarized clearly express that ratios were not merely fractions, or divisions between numbers, and that proportionality was a special kind of equality not

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to be confused with numerical equality. Leibniz, in the 1690's was the first to urge the use of the same symbols for fractions and for ratios and, consistently, the use of the sign « = » to express proportionality. Cogent as his reasons wew, «more than a century passed before his symbolism for ratio and proportion triumphed over its rivals» (77).

Concluding remarks

The medieval notion of denomination has been credited with bringing about, or at least facilitating, the arithmetization of ratios and proportionality (78). Yet in Pedro Núñez we have found an essentially fossilized theory of ratios, including denominations, side by side with the treatment of proportionality and ratios characteristics of the *abbaco* books and cossist algebras. This treatment included the free use of radical quantities and the free application of results originally in the arithmetical books of the *Elements* to ratios between magnitudes of any sort. That the two collateral ways of dealing with proportionality did not intermix is most interesting. To us it may appear a trivial step to say that the ratio of $\sqrt{2}$ to 1 has a denomination which is the result of dividing $\sqrt{2}$ by 1, namely $\sqrt{2}$. According to what we read in Núñez, however, that was not the case. Denominations remained a notion only applicable to rational ratios and tied to their names, while, from the operational, algorithmical point of view, denominations were superseded by the use of antecedent and consequent. More generally, it can be said that the medieval approach to ratios and proportionality was not assimilated by the *abbaco* and cossist books. Indeed I do not know of any other 16th-century algebra treatise, apart form Núñez's, containing traces of denominations (79).

From Jordanus Nemorarius to Oughtred ratios disappeared as a mathematical object to reappear as a notion of «common sense». In the modern understanding of proportionality no ratios other than the numerical ones are taken into consideration. Within the Euclidean conceptual framework, however, proportionality dealt with the general properties of ratios of different kinds of things. Pedro Núñez mentions, for instance, «numbers, lines, areas, volumes, angles, times, sounds, and movements» as different objects to which the theorems of Book V of the *Elements* apply (80). There is something other than a rhetorical bent in this long enumeration, for in a very true sense Núñez is saying that proportions apply in each case to the objects themselves, no to their

numerical measures (81). From Jordanus to Oughtred, therefore, the field in which proportionality applied narrowed. Yet one particular area of this field, numbers, broadened to encompass almost all the remaining areas. Because of the specialization to the numerical field, studying proportionality as a general tool became useless. In this rearrangement many of the properties of proportionality came to be considered easy consequences of arithmetical laws, and then incorporated to the lower levels of educational curricula.

As pointed out in the opening paragraphs of this paper, two different difficulties are noticeable in the way leading from the classical understanding of ratios and proportionality to the modern one: one concerns the status of the terms compared, and the other concerns the status of the ratios themselves. It is a historical fact that the two difficulties were solved in separate stages answering to different motivations. As shown above, a purely numerical understanding of the terms compared in ratios was achieved by the end of the 16th century. Central to this achievement was that within the algebraic tradition stemming from the *abbaco* books, Book VI of the *Elements* (on proportionality among geometrical magnitudes) came to be read «numerically», as if applying to numbers, and simultaneously results from the arithmetical books of Euclid were assumed to apply to geometrical magnitudes.

The second difficulty mentioned above cannot be fully unraveled here. Evidence stemming from the notational uses of the 17th century suggests that the evolution of the notions of ratio and proportionality is somehow related to the process of creation of the symbolic algebraic language (82). The first stage in the process of creation of modern mathematical symbolism ended by the mid 17th century, when the happy synthesis of symbolical solutions achieved by Descartes, with minor modifications, gained recognition and was widely adopted. The second stage ends in the central decades of the 18th century, when what we now call trascendent functions (logarithms, sine, and son on) received symbolic representation and their handling was incorporated into the algebraic language. D'Alembert, in his famous prefatory discours to the *Encylopédie*, plainly expressed the new status that algebraic language had gained by then. Geometry, he thought, was logically and epistemologically subordinate to algebra. Geometrical magnitudes may be used to illustrate algebraic results, but algebra is more general and abstract, and provides more efficient proofs than geometry does (83). At the same time the key concept of *function* unmistakably appeared, and some curves were reinterpreted as representation of functions.

At the beginning of the 17th century Galileo, without any kind of algebraic notation, used proportions to state relationships among continuously variable magnitudes. At the end of the same century Newton, who often represented proportions through algebraic notations, used proportions for the same purpose. Proportionality was used through the 17th century to express dependency, or functional relationships, much as it was used in medieval dynamics. Proportionality alone was then available to do this job (84).

It is justificable for us to conjecture, therefore, that ratios and proportionality gained a new and more «modern» status only in the 18th century, when a full-fledged concept of function appeared and proportionality lost its crucial role as *the* tool to express functional relationships.

NOTAS

(1) Λογος εστι δυο μεγεθων ομογενων η κατα πηλικοτητα ποια σχεσις in Euclid, *The Thirteen Books of The Elements*, 3 vols, T.L. Heath (ed.) 1956, New York, II, 115 (I have used Heath's translation). All references to the *Elements* will be made to this edition.

(2) ROSE, P. L. (1976): «Bartolomeo Zamberti's funeral oration for the humanist encyclopaedist Giorgio Valla», in C. H. Clough ed., *Cultural aspects of the Italian Renaissance*, New York, p. 302.

(3) ROSE, P. L. (1975): *The Italian Renaissance of Mathematics*, Geneve, chap 2, *passim*, but particularly p. 26-27.

(4) For instance, since the middle of the 15th century was known, through Regiomontanus, that Diphantus' manuscripts were of the highest interest, but they were translated only by the middle of the next century. See B. Hughes ed. (1967), *Regiomontanus on Triangles*, Madison, p. 11-8.

(5) ROSE: *Italian Renaissance of Mathematics*, p. 162-3.

(6) Quoted in DAVIS, M. D. (1977): *Piero della Francesca's Mathematical Treatises*, Ravena, p. 13.

(7) BAXANDALL, M. (1972): *Painting and Experience in Fifteenth-Century Italy*, Oxford, p. 86.

(8) VAN EGMOND, W. (1988): «How algebra came to France», in HAY, C. ed., *Mathematics from Manuscript to Print 1300-1600*, Oxford, p. 132. For the influence of the *abbaco* books on the development of algebra in the 16th century, see also Van Egmond's «The commercial revolution» (note 10); JAYAWARDENE, S. A. (1973), «The influence of practical arithmetics on the Algebra of Rafael Bombelli», *Isis*, 64, 510-23, and my (1973) «Les primeres àlgebres renaixentises», in MALET, A. & PARADÍS, J. (1984): *Els orígens i l'enseyament de l'àlgebra simbòlica*, Barcelona (chapter 5 in volume 1 of the Spanish translation, *Los orígenes del álgebra*, Barcelona, 1989).

(9) VAN EGMOND, W. (1980): «How algebra came to France», p. 134. The great majority of Italian *abbaco* treatises and commercial arithmetics are described in VAN EGMOND'S, W., *Practical mathematics in the Italian Renaissance: a catalogue of italian abbacus manuscripts and printed books to 1600* (Firenze: Istituto e Museo di Storia della Scienza).

(10) VAN EGMOND, W.: «How algebra came to France», p. 133. See also (1976) his «The commercial revolution and the beginnings of Western mathematics in Renaissance Florence, 1300-1500», Ph.D. Diss., Indiana University, *passim*.

(11) For a description of the main features of these books, see my «Les aritmètiques mercantils», in MALET, A. & PARADÍS, J. (1984): *Els orígens i l'ensenyament de l'àlgebra simbòlica*, Barcelona, chapter 4, volume 1, of the Spanish translation *Los orígenes del álgebra*, Barcelona, 1989). For the sources of these works, see VOGEL, K. (1971): «Leonardo da Pisa», in GILLISPIE, C. C. ed., *Dictionary of Scientific Biography*, IV New York, and PIOCHI, B. (1984): «Il Trattato di Paolo dell'Abaco», *Annali dell'Istituto e Museo di Storia della Scienza di Firenze*, 9: 21-39, which contains updated bibliographical references on primary sources now available in modern editions. Particularly interesting among them is the edition of the Treviso arithmetic (the first printed commercial arithmetic) in SWETZ, F. J. (1987): *Capitalism and Arithmetic*, La Salle, Illinois.

(12) JAYAWARDENE, S. A. (1976): «The "Trattato d'abaco" of Piero della Francesca», in CLOUGH, C. H. ed., *Cultural aspects of the Italian Renaissance* New York. The treatise is available in the modern edition of ARRIGHI, G. (1970) Pisa, to which all references are made.

(13) DELLA FRANCESCA: *Trattato*, p. 42.

(14) DELLA FRANCESCA: *Trattato*, p. 145.

(15) Books VII to IX, that is.

(16) BAXANDALL: *Painting and Experience*, p. 87.

(17) See, for instance, DELLA FRANCESCA'S, Piero (1942): *De prospectiva pingendi*, FASOLA, G. N.: ed. (Florence).

(18) PANOFSKY, E. (1955): *The Life and Art of Albrecht Dürer*, Princeton, 1, p. 262-7, plates 312, 316, 317, 319, 322.

(19) On medieval versions of the *Elements*, see MURDOCH, J. E. (1963): «The Medieval Language of Proportions», in CROMBIE, A. C. ed., *Scientific Change*, London; «The Medieval Euclid», *Revue de Synthèse*, 3rd ser., 89, 67-94; MAHONEY, M. S. (1978): «Mathematics», in LINDBERG, D. C. ed., *Science in the Middle Ages*, Chicago, and MOLLAND, A. G. (1968-69): «The Geometrical Background to the Merton School», *British Journal for the History of Science*, 4, 108-25.

(20) Other variations were introduced in the medieval Latin versions of the *Elements*, but they did not embody important alterations, as far as the mathematical content is concerned; see MURDOCH: «The Medieval Euclid», p. 74-80. Proposition 12, Book VI, provides the construction of the fourth proportional to three given rectilinear segments: «Two three given straight lines to find a fourth proportional» (cf. Heath ed., *Elements*, II, 215). The proposition disappeared from Campanus' edition, but an spurious postulated appeared instead, which assured the existence of a fourth proportional term in general: «By as much as some one quantity is to another quantity of the same genus, so much is a third *[quantity]* to some fourth of the same genus» (cf. MURDOCH: «The Medieval Language of Proportions», p. 250; Murdoch points out that the reference to «the same genus» seems to be a later addition). The addition of such a postulate is not without justification. From the 17th century on many editors and commentators of Euclid, in-

cluding Clavius. Saccheri, Simson and De Morgan, agreed that this axiom filled a logical lacuna in the *Elements* (*Elements*, II, 170-4). What the axiom postulates is tacitly used in Proposition V-18, which proves that $(a+b):b::(c+d):d$, if $a:b::c:d$, and also in Proposition XII-2, which demonstrates that the circles are as the squares of the diameters. Once the axionn was explicitly postulated, Proposition VI-12 became a redundant result, since Proposition VI-2, proved that in any triangle ABC, the segment DE parallel to BC cuts the sides AB and AC proportionally, and conversely (ibíd., II, 195-5).

(21) *The Elements*, II, 114. In modern terms, $a:b::c:d$ when for any integer numbers m and n, ma is greater than, equal to or, less than nb if and only if mc is greater than, equal to, or less than nd.

(22) *Ibíd*.

(23) Murdoch: «The Medieval Language of Proportions», p. 256.

(24) Busard, H. L. L. (1970): «Die Traktate *De proportionibus* von Jordanus Nemorarius und Campanus», *Centaurus*, 15, 193-227; quotation comes from page 205.

(25) See Murdoch, J. E.: «The Medieval Language of Proportions». There is no incontrovertible evidence to assign an specific source to Jordanus' definition of denomination. Compare my interpretation with Mahoney's, in «Mathematics», p. 163.

(26) In the example above, «super» indicates that the ratio is of the kind $1+a/b$; «bipartiens» indicates that a is two: and «thirds» tells the reader that b is three. On the medieval nomenclature of ratios, see Mahoney, S.: «Mathematics».

(27) Murdoch: «The Medieval Euclid», p. 80. Campanus included this definition as the 20th of Book VII. It had already appeared as definition II-9 in Jordanus' *Arithmetica* (Paris, 1495).

(28) Murdoch: «The Medieval Language of Proportions», p. 257-9.

(29) The determination of a fourth term in Proposition VI-12, for instance, is solved by a configuration associated to the so-called Thales' Theorem, which establishes the proportionality of the segments determined on two sides of a triangle by a line parallel to the third side (*Elements*, II, 215-6; cf. note 20 above).

(30) *De numeris datis*, which somehow reminds the reader of Diophantus' *Arithmetic*, contains 3 definitions and more than a hundred propositions, or rather arithmetical problems. Proposition II-14a, for instance, runs thus: Let from two given numbers be subtracted numbers which are in a given ratio and let the product of the remainders be known, then the numbers are given, cf. Hughes, B. ed and trans. (1981), *Jordanus de Nemore De numeris datis*, Berkeley, p. 75. Proposition II-14a admits of the translation, numbers x,y such that $x/y=d$ and $(a-x)(b-y)=c$ can always be determined.

(31) *Ibíd*., p. 70.

(32) In the *Elements* Proposition VII-19 characterizes proportionality by means of the equality of the cross products (a, b, c, d are proportional terms if and only if $ad=bc$). Along with the rules that determine the mean proportional as the square root of the extreme terms, the *Elements* also sets forth rules to determine proportional means of higher order as roots with higher exponents.

(33) See, for instance, Propositions IX-18 and IX-19 of the *Elements*.

(34) To be sure, Jordanus used abundantly of the notion of denomination in this and other works of his. Thus, for instance, Proposition 2 from Book II teaches how to determine a number when the ratio it helds with a given number is known. In orden to have the antecedent, it suffices to multiply the known number by the denomination *(sic)*, which means that given the denomination of (A:B), say, superbipartiens thirds, and given B, then the multiplication of B by $1+2/3$ yields A (*De numeris datis*, Hughes ed., p. 70). Pro-

position II-3 says that 1 divided by the denomination of (A:B) gives us the denomination of (B:A) (*ibíd.,* p. 71). Proposition II-8 states that the sum of the denominations of thre ratios that several numbers held to a given number yields the ratio that all the former taken together bear to the given number. This may admit the translation (A:D) + (B:D) + (C:D) = (A + B + C:D) = (*ibíd.,* p. 72). Jordanus also stated elsewhere that the composition of two ratios whatever is found through the multiplication of their denominations (Busard, «Die Traktate *De proportionibus*», p. 205).

(35)   On the denominations of irrational ratios, see. GRANT, E. (1986): *Nicole Oresme «De proportionibus proportionum» and «Ad pauca rescipientes»,* Madison, p. 31-5, and MAHONEY, M. S.: «Mathematics». According to Grant, Bradwardine's mention of denominations of irrational ratios are among the first we know of. Those ratios are called irrational, says Bradwardine, which are not immediately denominated by a number, but only mediately. Any irrational ratio is immediately denominated by a ratio which es immediately denominated by a number. («Secundum vero gradum illa tenet quae irrationalis vócatur, quae non immediate denominatur ab aliquo numero, sed mediate tantum (quae immediate denominatur ab aliqua proportione, quae immediate denominatur a numero:...)»; see CROSBY, H. L. Jr (1961): *Thomas of Bradwardine. His «Tractatus de Pro-portionibus»,* Madison, p. 66). Although Bradwardine set forth the definition of equal ratios based in the equality of denominations, he did not showed himself concerned in explaining how this definition was to be used with irrational ratios and their *mediate* denominations. As a matter of fact, Bradwardine did little use of mediate denominations in general. Grant points out that, in all probability, Bradwardine's use of denominations stemms from a source which he did not understand.

(36)   GRANT: *Nicole Oresme,* p. 31-3.

(37)   *Ibíd.,* p. 38 and 49.

(38)   As is well known, Oresme's ultimate purpose was to undercut the scientific basis of astrology. Once he established that it is unlikely that two unknown ratios be commensurable, he argued that «it is most unlikely that the as yet unknown exact ratios of planetary motions will be commensurable. [Since] astrology rests on the commensurability of those motions, ... astrology is at best scientifically suspect» (MAHONEY: «Mathematics», p. 168).

(39)   GRANT: *Nicole Oresme,* Proposition III-2, p. 225.

(40)   *Ibíd.,* p. 39.

(41)   MURDOCH: «The Medieval Language of Proportions», p. 261. Oresme certainly knew of the uses that denominations of rational ratios have from an algorismic point of view, for he devoted one of this works, *Algorismus proportionum,* to deal with operations between ratios performed through their denominations; see GRANT: *Nicole Oresme,* p. 315.

(42)   STEVIN, S. (1958): *The principal works of Simon Stevin* (5 vols. Amsterdam) II, 474 and 577. On Núñez's *Libro de Algebra* see BOSMANS, H. (1908): «Sur le Libro de algebra de Pedro Núñez», *Bibliotheca mathematica,* 8, 154-69, and my «L'aritmètica i l'àlgebra a la Península Ibèrica al segle XVI», in MALET, A. & PARADÍS, J.: *Els orígens i l'ensenyament de l'àlgebra simbòlica* (Barcelona, 1984) (chapter 9 in volume 1 of the Spanish translation, *Los orígenes del álgebra,* Barcelona, 1989).

(43)   *Libro de Algebra,* p. 66^r.

(44)   *Ibíd.,* p. 75 ^{V}; the notation (A:B) is not Núñez's.

(45)   *Ibíd.*

(46)   *Ibíd.,* p. 76 ^{V}.

(47) *Ibíd.*, p. 76 ᵛ - 77ʳ.

(48) *Ibíd.*, p. 85ʳ.

(49) *Ibíd.*, p. 85ʳ and 83ʳ.

(50) *Ibíd.*, p. 84 ᵛ - 85ʳ. Núñez underscores the distance between his two methods to compose ratios by grounding them on different proofs; he sends the reader to his book on Oronce Finné's quadrature of the circle for a proof of the method which uses the product of denominations.

(51) *Ibíd.*, p. 75ʳ.

(52) Apparently this is not Oresme's notion of immediate and mediate denominations, at least not as it was intepreted by Grant. See above, n. 35.

(53) Núñez: *Libro de Algebra*, p. 99 ᵛ.

(54) This is Proposition VI-16 from the *Elements.*

(55) *Ibíd.*, Proposition VI-12.

(56) Núñez: *Libro de Algebra*, p. 104 ᵛ - 105ʳ.

(57) *L'Arithmetique*, p. 1-5 and 8-10; references are made to *Les Oeuvres Mathematiques de Simon Stevin* (Leyde, 1634). For an account of Stevin's algebra and arithmetic, see my *Simon Stevin* (volume 3 of *Los orígenes del álgebra*, Barcelona, 1990). On Stevin's grounding of a continuum of arithmetical magnitudes on a geometrical continuum, see Klein, J. (1968): *Greek Mathematical Thought and the Origin of Algebra*, Cambridge, Mass, p. 186-97, and D. Struik's account in Stevin, S. (1958), *The principal works of Simon Stevin*, Amsterdam, IIB, p. 459-61.

(58) *L'Arithmetique*, p. 15.

(59) Stevin, however, did not achieve a complete unification of rational and radical numbers. Thus, according to the kind of magnitudes dealt with, Stevin needs two unrelated propositions to determine the fourth proportional to three given magnitudes. Given three *rational* numbers, *a, b, c*, Stevin did not feel the need to justify that $x = b.c/a$ was the fourth proportional term; he explicitly restricted the field of applicability of this rule to rational numbers *(L'Arithmetique*, Problem XIV, p. 24). In order to justify this rule when applied to *radical* numbers, Stevin turned to the propositions that set forth how to find the ratio between two radical numbers (*ibíd.*, p., 51).

(60) *La Practique d'Arithmetique*, p. 177, in *Les Oevres Mathematiques de Simon Stevin.*

(61) On Viète's contribution to the new understanding of mathematics in the 16th century, see Mahoney, M. S. (1973): *The mathematical career of Pierre de Fermat*, Princeton, ch. 2, *passim*, and Klein, J.: *Greek Mathematical Thought.* The *Isagoge* is easily available through the 1970 reprint of the 1646 Leiden edition of Viète's *Opera mathematica.* Translations come from Viète (1983): *The Analytic Art*, Witmer, T. R. ed. and tr., Kent (Ohio).

(62) Viète, F.: *The Analytic Art*, p. 14-5.

(63) *Ibíd.*, p. 15.

(64) Morse, J. (1981): «The Reception of Diophantus' Arithmetic in the Renaissance», Ph. D. Diss., Princeton University, p. 155-9.

(65) *Opera mathematica*, p. 13.

(66) On Oughtred's *Clavis*, see Bosmans, H. (1910-1911): «La première edition de la "Clavis Mathematica» d'Oughtred», *Annales de la Société Scientifique de Bruxelles*, 35, 24-78, and Cajori, F. (1916): *William Oughtred*, Chicago. In them one can still fin the last echoes of the polemic raised by Wallis' chauvinist comparison of Oughtred's and Harriot's works with Descartes' *La Géometrie*, and by his attacks against the latter as a

plagiarist fo the English mathematicians. Bosmans and Cajori also specify the main varia-tions introduced in the *Clavis* in the second and subsequent editions. Quotations come from the 1631 London, edition, *Arithmeticae in numeris et speciebus institutio: ..., quasi clavis est*, but are given in their English version in the 1647 edition, *The Key of the Mathematics new forged and filed.*

(67)   *Clavis* (1647), p. 16.

(68)   *Ibíd.*

(69)   *Ibíd.*

(70)   *Ibíd.*

(71)   *Ibíd.*

(72)   See CAJORI, F. (1928): *A History of Mathematical Notations*, 2 vols, Chicago, I, p. 275-95.

(73)   In fact he used «a.b:c.d» instead of the expression given above. In the 1650's WING, V., also an Englishman, substituted the semicolon for the dots (CAJORI; *History of Mathematical Notations*, I, p. 275).

(74)   HERIGONE, P. (1634-1637): *Cursus mathematicus*, 5 vols. Paris. This mathematical encyclopedia containing, among many other things, the first printed account of the sine law of refraction was very influential on the continent. The example given here comes actually from *Le Supplement du Cours Mathematique* (Paris, 1642), unnumbered page following the title page.

(75)   DESCARTES, R. (1966): *Oeuvres*, ADAM, C. & TANNERY, P.: eds., 10 vols., Paris, X, p. 241.

(76)   DESCARTES did not use this, or any other, notation at all in his 1637 *Geométrie*, although he used it again in a letter written in 1638 (*Oeuvres*, II, 171).

(77)   CAJORI: *History of Mathematical Notations*, I, p. 295.

(78)   This view has been more fully defended by MAHONEY, M. S.: «In short, via the procedure of denomination, ratios came to be manipulated by the arithmetic of fractions. The short-term result was an arithmetization of the theory of ratio and proportion that evaded or ignored the subtler aspects of that theory» («Mathematics», p. 164).

(79)   The fossil character of the denomination-based approach to ratios, the fact that is survived almosty whithout changes for 200 years, as Nuñez's pages evince, bespeaks an intellectual isolationism which hardly corresponds to a notion that would have modified other notions.

(80)   *Libro de Algebra*, p. 70^r.

(81)   The distance separating conceptually objects and their measures is neatly pre-sent in Núñez's discussion of the paradox concerning the so-called angle of contingency, or angle between a straight line tangent to a circle and the circle itself. Núñez considers this angle different from zero, although he accepts as a matter of course that there is no rectilinear angle smaller than the angle of contingency. He does not even justify that the angle of contingency is not nil: it is a visual truth. Furthermore, angles of contingen-cy can be compared one to another. For example, two circles tangent at a point A deter-mine an angle of contingency which is twice the angle between one of the circles and the straight line tangent to the circles at the same point A. This puzzle origins, accor-ding to Núñez, in this, that the quantities here compared are not of the same nature, since we can not overcome a rectilinear angle multiplying the angle of contingency by a number. Núñez emphasizes, finally, that the paradox could never arise in dealing with numbers, since any number, no matter how small, surpasses a given number, no matter how large, when the small is adequately multiplied *(Libro de Algebra,* p. 66 ^{v} - 69 ^{v}).

(82) On the 17th-century debates concerning ratios and proportionality, see E. Sylla's insightful discussion, «Compounding ratios. Bradwardine, Oresme, and the first edition of Newton's *Principia*», in MENDELSOHN, ed. (1984), *Trasformation and Tradition in the Sciences*, Cambridge.

(83) D'ALEMBERT, J. (1984): *Discurso preliminar de la Enciclopedia*, Madrid, p. 50-1 and 57ff.

(84) MAIER, A. (1982): «The concept of function in Fourteenth-Century Physics», in SARGENT, S. D., ed., *On the Treshold of Exact Science*, Philadelphia, *passim*. Compare with BOYER, C. B. (1946): «Proportion, Equation, Function: Three steps in the development of a concept», *Scripta Mathematica*, 12, 5-13.